It is worthy of note that the modern fractional exponent was first introduced by Newton in the announcement of his Binomial Theorem, invented by him some time before 1669. Newton used also negative fractional exponents. In November, 1676, Leibniz collected some of his results on a sheet of paper; he uses here the notation[1] x^{-3}, $x^{-\frac{1}{3}}$. It should be observed also that the use of *literal* exponents is suggested in Newton's form for the binomial theorem as given above, and that literal exponents came to be generally used in the latter part of the seventeenth century by Newton, Leibniz and their followers. Perhaps the earliest occurrence of literal exponents is in Wallis's *Mathesis universalis*, Oxford, 1657, where a few expressions like $\sqrt[d]{R^d} = R$, $AR^m \times AR^n = A^2R^{m+n}$ have been noticed.[2]

The theory of exponents, involving positive, negative and fractional values, is explained and freely used in the much respected and widely read work, entitled, *Analyse demontrée*, by Charles Reyneau, Paris, 1708. The theory is explained in the introduction to the first volume. This is done because the treatises on algebra then in use did not usually contain it. Reyneau uses these words: "Le seul calcul qui n'est pas expliqué dans les Traités d'Algebre dont on vient de parler, est celui des exposants des puissances."[3] In deriving rules for differentiation,[4] Reyneau passes from $x^x = a$ to $xlx = la$, and from $x^{x^x} = y^{y^y}$ to $x^x lx = y^y ly$.

The interesting question arises, when and where did the union between the exponential and logarithmic concepts take place? It did not occur until the eighteenth century. As is quite proper, there was quite a long courtship. It goes back to the time of Wallis. In the twelfth chapter of his *Algebra*, 1685, Wallis develops the theory of logarithms, beginning with two progressions 1, 2, 4, 8, $\cdots$ and 0, 1, 2, 3, $\cdots$. He then generalizes by taking

$$1 \cdot r \cdot rr \cdot r^3 \cdot r^4 \cdot r^5 \cdot r^6 \text{ etc.}$$
$$0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \text{ etc.}$$

and remarks that "*these exponents they call logarithms*, which are artificial numbers, so answering to the natural numbers, as that the addition and subduction of these answers to the multiplication and division of the natural numbers." And yet, Wallis does not come out, resolutely, with the modern definition of a logarithm, and use it.

A similar point of view was reached[5] by John Bernoulli I in a letter of May, 1694, addressed to Leibniz. Bernoulli discusses an "ideam novi $\cdots$ *calculi percurrentis*," a terminology later discarded in favor of calculus of "exponential quantities." He speaks of the construction of exponential curves $x^x = y$ by

[1] C. I. Gerhardt, *Der Briefwechsel v. G. W. Leibniz mit Mathematikern*, Bd. I, Berlin, 1899, p. 230.

[2] G. Eneström, *Bibliotheca mathematica*, 3d S., Vol. 9, 1908–1909, p. 329.

[3] Ch. Reyneau, *Analyse demontrée*, Vol. I, Paris, 1708, p. xvii.

[4] *Analyse demontrée*, Vol. II, 1708, p. 806.

[5] Got. Gul. Leibnitii et Johan. Bernoullii *Commercium philosophicum et mathematicum*, T. I, 1745, p. 8.

means of the ordinary logarithmic curve which he says is itself a curve of that type, having the equation $a^x = y$. Bernoulli assumes the logarithmic curve to be drawn and uses the graph for plotting the equation $x^x = y$. He assumes a value x_1, then measures off the ordinate $\log x_1$ on the logarithmic curve and geometrically constructs the product $x_1 \log x_1 = \log y_1$. Finally he finds, again by the aid of the logarithmic curve, the antilogarithm y_1. This y_1, together with the assumed value of x_1, yields a point on the curve $x^x = y$. Thus the curve can be constructed by points. From our point of view, the interest of this process lies in the fact that the logarithmic curve and therefore the logarithm itself, is connected with the equation $a^x = y$. Here x is looked upon as the logarithm of y. Bernoulli makes no mention here of arithmetic and geometric progressions. His procedure involves the modern definition of a logarithm, which, however, he does not explicitly state. The process shows that Bernoulli passed from $x^x = y$ to $x \log x = \log y$, though he did not actually write down this last equation. In June, 1694, Leibniz sent J. Bernoulli a letter in reply, in which he writes[1] both $x^x = y$ and $x \log x = \log y$. We see from the above that Leibniz and J. Bernoulli had a grasp at this time of the exponential *function*.

II. From Leibniz and John Bernoulli I to Euler. 1712–1747.

Unsuccessful attempts to create a theory of logarithms of negative numbers.

In the eighteenth century the tendency to take rules derived only for a special case and apply them to more general cases became more pronounced than it had been. It is a tendency which in the nineteenth century came to be called the "principle of the permanence of equivalent forms" or, better still, the "principle of the permanence of formal laws." To-day, we look upon these extensions as things we are at liberty to do or not to do, as we may please. If we find it most convenient in a given research to reject negative and complex numbers and confine ourselves to positive numbers, we may do so, but we are expected to state our position clearly, then to maintain it. In the seventeenth and eighteenth centuries it was not clearly felt that, logically, one had this freedom to extend or to limit the number concept. Negative numbers came to be used freely, yet this extension of the domain was done with misgivings, which show themselves in the names applied to them, such as *false* or *defective* numbers, *numeri ficti*. Still more pronounced was the feeling of discomfort toward bi or $a + bi$; such numbers were called *imaginary, impossible*. It was felt that the validity of negative and complex numbers should be *proved*, not *assumed;* that the rules of operation with such numbers was a matter requiring *demonstration*. Hence the eighteenth century mathematicians, including even men of the type of Laplace, tried to prove the rule of signs in the multiplication of two negative numbers. The "proofs" given were futile; they rested on a syllogism without

[1] Leibnitii et Bernoullii *Com. phil. et math.*, I, p. 10.

a major premise. This difference in the point of view must be borne in mind in the history of the extension of the logarithmic concept to negative and complex numbers. It will help to explain how it was that the controversy on this subject lasted for a whole century and reached well into the nineteenth century.

Several of the eighteenth century mathematicians of the first rank, particularly Euler, used imaginaries freely; some other mathematicians looked upon the imaginaries with suspicion. Here and there an eighteenth century mathematician declaims loudly against the use both of the negative and the imaginary.[1] Interesting is the language used by Leibniz in 1702. He speaks of the imaginary factors of $x^4 + a^4$ as "an elegant and wonderful recourse of divine intellect, an unnatural birth in the realm of thought, almost an amphibium between being and non-being."[2] Most wonderful was the result reached[3] in 1702 by John Bernoulli[4] (1667–1748). He explained the transformation of the differential $adz \div (b^2 + z^2)$ into $-adt \div 2bt\sqrt{-1}$, by means of the relation $z = (t-1).\ b\sqrt{-1} \div (t+1)$ and thereby showed that the integral can be expressed as an arctangent and also as a logarithm. In this manner he pointed out a relation between the logarithm of an imaginary number and the arctangent. This *logarithme imaginaire*, as John Bernoulli called it, was so novel and so foreign to the thought of the time that it caused little comment.

In a letter of the same year (1702), dated June 24 and addressed to John Bernoulli, Leibniz speaks of imaginary logarithms in connection with the problems of integration.[5] There is danger of attaching too much importance to passages of this sort. To Leibniz and J. Bernoulli an imaginary often meant simply non-existence. If logarithms of imaginary numbers were believed to exist, nothing is here brought out as to the nature of such logarithms.

The controversy on logarithms which agitated mathematicians for more than a century did not originate primarily in discussions of imaginary number, but rather in discussions of negative number. Are negative numbers less than nothing? If they are, then in a proportion $1 : -1 = -1 : 1$, the greater number is to the less, as the less is to the greater — an impossibility. This matter was discussed by many writers, including Leibniz, Newton, D'Alembert, Maclaurin, Rolle and Wolf. Leibniz published a paper on this subject in 1712.[6] He considered the above proportion impossible in fact, but maintained that such proportions may be used with the same advantage and safety with which other inconceivable quantities are used. Leibniz said that a ratio may be considered imaginary, when it has no logarithm. The ratio $-1 \div 1$ has no logarithm; for, there would result $\log(-1/1) = \log(-1) - \log 1 = \log(-1)$.

[1] See Cantor, *op. cit.*, Vol. 4, 1908, pp. 79–90.

[2] Leibniz, *Werke*, Ed. Gerhardt, 3. F., Bd. V, 1858, Berlin, p. 357: *Itaque elegans et mirabile effugium reperit in illo Analyseos miraculo, idealis mundi monstro, pene inter Ens et non-Ens Amphibio, quod radicem imaginariam appellamus.*

[3] Joh. Bernoulli, *Opera*, Vol. I, Laus. et Genevæ, 1742, p. 399.

[4] For the sake of distinction, this John Bernoulli is frequently designated as John Bernoulli I.

[5] *Leibnitii et J. Bernoullii Commerc. Phil. et math.*, T. II, 1745, p. 81.

[6] *Acta Eruditorum*, 1712, pp. 167–169; *Werke*, 3. F., Bd. V, Halle, 1858, pp. 387–389.

Leibniz declared that -1 has no real logarithm; such a logarithm could not be positive, for a positive logarithm corresponds to a number larger than 1; the logarithm could not be negative, for a negative logarithm corresponds to a positive number, less than unity. The only alternative remains, therefore, to declare the logarithm of -1 as not really true, but imaginary. He arrives at the same conclusion from the consideration that if there really existed a logarithm of -1, then half of it would be the logarithm of the imaginary number $\sqrt{-1}$, a conclusion which he considered absurd. We notice in these statements of Leibniz a double use of the term imaginary: (1) in the sense of non-existent, (2) in the sense of a number of the type $\sqrt{-1}$.

On March 16 of this year, before the appearance of the article, Leibniz mentioned the subject in a letter to John Bernoulli. That letter opened up a friendly controversy between the two men on the logarithms of negative and imaginary numbers. In their correspondence they debated this question for sixteen months. At that time they were the only ones interested in this question; in fact, they were the only ones to whom the problem of the existence or non-existence of logarithms of negative numbers had occurred. The controversy opens up a number of most interesting points and gives an insight into the algebraic concepts of the time as could not be obtained readily in any other way. For brevity let $+n$ or $+x$ indicate a "positive number," $-n$ or $-x$ a "negative number," $\log(+n)$ the "logarithm of a positive number," $\log(-n)$ the "logarithm of a negative number," i or in, an "imaginary number." The following is a synopsis of the correspondence:[1]

March 16, 1712. Leibniz to J. Bernoulli. This is the letter already referred to. L. says that $-1/1$ is imaginary, since it has no logarithm.

May 25, 1712. J. Bernoulli to Leibniz. B. rejects L.'s proof that the ratio $1:-1$, or $-1:1$ is imaginary, for the reason that $-x$ has a logarithm. We have $dx:x=-dx:-x$; hence, by integration, $\log x=\log(-x)$. The logarithmic curve $y=\log x$ has therefore two branches, symmetrical to the axis Y, just as the hyperbola has two opposite branches.

June 30, 1712. Leibniz to J. Bernoulli: L. repeats his argument that $\log(-2)$ does not exist; for, if it did, its half would equal $\log\sqrt{-2}$, an impossibility. The rule for differentiating, $d\log x=dx:x$, does not apply[2] to $-x$. In the logarithmic curve $y=\log x$, x cannot decrease to 0 and then pass to the opposite side, since the curve cannot cut the Y axis, which is asymptotic to it.

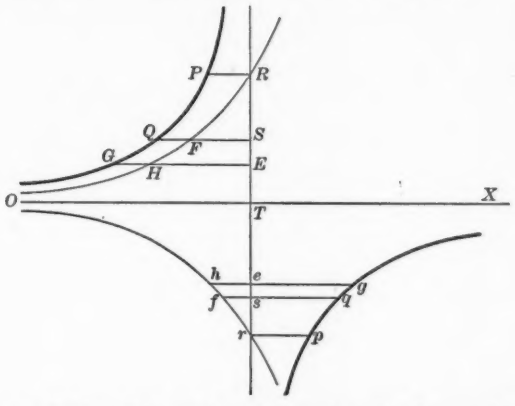
August 13, 1712. J. Bernoulli to Leibniz: The argument that $\log(-2)$ does not exist, because $\log\sqrt{-2}$ does not exist is invalid. I deny that $\log(\sqrt{-2})$ is half of $\log(-2)$, even though it be true that $\log\sqrt{2}=\frac{1}{2}\log 2$. The difference is that $\sqrt{2}$ is a mean proportional between 1 and 2; $\sqrt{-2}$ is not a mean proportional between -1 and -2. Just as $\log\sqrt{1\times 2}=\frac{1}{2}\log 2$, so is $\log\sqrt{-1\times -2}=\frac{1}{2}\log(-2)$. That is, $\log\sqrt{2}=\frac{1}{2}\log 2=\frac{1}{2}\log(-2)$. In passing from $+x$ to $-x$ in a curve it is not necessary that the curve cut the y-axis. Witness the conjugate hyperbola whose abscissas are common to $+$ and $-$ ordinates, but never to vanishing ones. In $y=\log x$, one branch of the curve passes into the other at infinity, when $x=0$, in the same manner as in the conchoid of Nicomedes and other curves.

[1] Got. Gul. Leibnitii et Johan. Bernoullii *Commercium Philosophicum et Mathematicum*, Tomus Secundus, Lausannæ et Genevæ, 1745, pp. 269, 276, 278, 282, 287, 292, 296, 298, 303, 305, 312, 315.

[2] *Sed hæc regula, quod differentiale divisum per numerum dat differentiale Logarithmi, et quævis alia de Logarithmorum natura et constructione non habet locum in numeris negativis, ut reperies, ubi demonstrare voles.*

Sept. 18, 1712. Leibniz to J. Bernoulli: Logarithms are numbers in arithmetic progression, corresponding to numbers in geometric progression, of which one number may be 1 and another may be any positive number. Assume $\log 1 = 0$ and $\log 2 = 1$. In the geometric progression thus limited, $-n$ can never be obtained, no matter how many third proportionals are formed. In the series 1, 2, 4, the mean proportional between 1 and 4 is both $+2$ and -2. But -2 cannot be in the same geometric progression which contains $+2$; that is, no value of e makes $-2 = 2^e$ or $e = \log(-2)$; hence there is no logarithm of -2, and a curve $e = \log x$ that is satisfied by $x = 1$ and $x = 2$, cannot be satisfied by $x = -2$. *Otherwise thus:* If -2 has a logarithm, then the half of this logarithm exists and is the logarithm of $\sqrt{-2}$. But $\sqrt{-2}$ is an impossible number; hence, half of $\log(-2)$ is impossible, and the whole, or $\log(-2)$ is impossible. *Another point:* In logarithmic theory, n^e or $\sqrt[e]{n}$ is represented by $\log n \cdot e$, or $\log n : e$; nn or $n : n$ is represented by $\log n + \log n$; n is represented by $\log n$; by what is $-n$ represented? There is no mode of representation below the ones already named. *Again:* Granting for the moment that $\log(-2)$ exists, it follows that $\log \sqrt{-2}$ is half of it, for $\sqrt{-2}$ is the mean proportional between $+1$ and -2; hence, $\log \sqrt{-2} = (\log 1 + \log(-2)) : 2 = \frac{1}{2} \log(-2)$.

Nov. 9, 1712. J. Bernoulli to Leibniz: B. says that he sees nothing in the last letter which proves the impossibility of $\log(-n)$. He admits that there is no transition from a (geometric) series of positive terms to one of negative terms, and so $\log(-n)$ does not exist in this case.[1] But negative numbers determine their own peculiar series starting with -1, instead of $+1$ Thereby the same logarithmic properties follow for $-n$ as for $+n$. He reiterates that



$\log n = \log -n$. To show that $y = \log x$ has two branches, he uses the rectangular hyperbola $PQGpqg$ and lets SF and EH be proportional to the hyperbolic areas $RSQP$ and $REGP$. Let PR and GE be constants and FS a variable. As S touches T, FS is infinite and the area is infinite. Now keeping to the same law of generation of the curve RFH, let the point S proceed to e (for what can hinder this?). The area upon Re is partly $+$ and partly $-$, and equal to EP, when $TE = Te$. We have then $EH = eh$. Similarly, if $Ts = TS$, then $sf = SF$. Thus arises the branch hfr which, with HFR, constitutes the one logarithmic curve, just as the two branches of the hyperbola constitute one curve. If $TR = +1$, $Tr = -1$, $TS = +n$, $Ts = -n$, then $SF = \log n$, $sf = \log(-n)$. As $SF = sf$, we must have $\log n = \log(-n)$.

Jan., 1713. Leibniz to J. Bernoulli: Assuming $2^e = x$, if $x = 1$ then $e = 0$, if $x = 2$ then $e = 1$. When $x = -1$, e cannot be assigned.

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[1] *Hoc unum efficis omnibus Tuis argumentis, ut ostendas non dari transitum ex serie numerorum affirmativorum in seriem negativorum, hoc est, assumpta unitate (nempe +1) pro initio seriei numerorum, nullum numerum negativum ex illa serie inveniri posse, adeoque nullos eorum logarithmos hoc casu existere; quod quidem non nego. Sed hoc non impedit, quominus numeri negativi suam peculiarem constituant seriem, assumta pro eorum initio unitate negativa, (nempe −1).*

Feb. 28, 1713. J. Bernoulli to Leibniz: If in $v^e = x$ we assume $x = 2$ and $e = 1$, also $x = 1$ and $e = 0$, then truly e cannot be assigned when $x = -1$. As these assumptions are arbitrary, change them so that $x = -1$ when $e = 0$, then e can be assigned for any $-x$.

April 26, 1713. Leibniz to J. Bernoulli: You say that my values for e and x in $2^e = x$ are arbitrary. You let $x = -1$ and $e = 0$. Mine is the most natural. Aside from that consider *firstly* that we cannot have both $\log n$ and $\log(-n)$, for if $\log(-1) = 0$, then $\log(-1)^2 = \log 1 = 2 \times 0 = 0$, and $\log \sqrt{-1} = 0/2 = 0$. That is, one and the same logarithm is obtained for $+1$, -1, and i. *Secondly:* On your assumptions 2^0 has an infinite number of meanings, for 2^0 will equal -1, also $+1$, $\sqrt{-1}$, $\sqrt[4]{-1}$, $\sqrt[8]{-1}$, etc. Unless 2^0 is many-valued, these must all be equal to each other. If $2^0 = +1$, then 2^0 is single-valued and no such difficulty arises. *Thirdly:* If $x^e = -2$, $x^{2e} = +4$, but this transition from $-n$ to $+n$ you yourself reject. *Fourthly:* If $\log(-n)$ is real, then $\log \sqrt{-n}$ is real; hence, impossible numbers would have possible logarithms. The assumption that only $+n$ have logarithms avoids this trouble. *Fifthly:* From the beginning you have admitted that we cannot have $2^e = 1$ and $2^e = -1$ at the same time. But if you put $2^0 = -1$, then $(2^0)^2 = 2^0 = +1$. Hence $2^e = 1$ and $2^e = -1$, for $e = 0$. This is contrary to your admission.

All three things show that your hypothesis concerning $\log(-n)$ is unnatural, useless, and inadmissible. I have shown elsewhere that proportions cannot be formed, involving $-n$. If it be true that the two fractions $+1/-1$ and $-1/+1$ are equal, observe that fractions are not the same as ratios. It is evident from all this that the very foundations of things analytical have been neglected thus far.[1]

June 7, 1713. Bernoulli to Leibniz: What do you understand by *natural?* If that is natural which conforms with usage, then $\log(-n)$ is less natural than $\log(+n)$. The first of your five objections to $\log(-n)$ is that some $+n$, $-n$, *in* would have the same logarithms. I admit only that $\log(+n) = \log(-n)$. The half of any logarithm is not necessarily the logarithm of the square root; it is rather the logarithm of the mean proportional between $+1$ and $+n$, or -1 and $-n$. The mean proportional between -1 and -1 is $\sqrt{-1 \times -1} = +\sqrt{+1}$ or $-\sqrt{+1}$. There is nothing absurd in this. *Secondly:* I deny that $2^0 = \sqrt{-1} = \sqrt[4]{-1}$, etc. As just explained, $2^0 = \sqrt{-1 \times -1}$ and $= \sqrt{-1 \times -1 \times -1 \times -1}$, etc. All these radicals equal $\sqrt{+1}$ or $\doteq 1$. There is no discord in these results. *Thirdly:* You say that if $x^e = -2$, then $x^{2e} = +4$. The logarithmic curve shows this to be untrue. Twice $\log(-n)$ is not $\log n^2$. The third proportional of $-n$ is obtained from $-1 : -n = -n : x$. Hence if $x^e = -2$, then $x^{2e} = -2 \times -2 \div -1 = -4$. Consequently there is no crossing from $-n$ over to $+n$. *Fourthly:* My definition of the mean proportional of $-n$ does not lead to the absurd result that *in* has a possible logarithm. *Fifthly:* If $2^0 = -1$, then $2^{2 \cdot 0}$ is not $= +1$, but to $-1 \times -1 : -1 = -1$. Hence the absurd result does not follow that 2^0 is at the same time $+1$ and -1.

June 28, 1713. Leipniz to J. Bernoulli: I have no time to disprove your objections to my doctrine which makes $\log i$ impossible, the double of impossibles impossible, $\log n$ the double of $\log \sqrt{n}$. If you assume logarithms in which this is not so, that is nothing to me. I call the more *natural*, not that which is more customary, but that which is nearer to nature and the more simple.

July 29, 1713. J. Bernoulli to Leibniz: You do not deny that the assumption $+1$ is arbitrary, and that -1 is permissible. According to the latter, $\log(-1) = 0$. From this follows all I have previously said about $\log(-n)$.

It is easy to see that Leibniz and J. Bernoulli could not come to an agreement on $\log(-n)$, as long as they did not agree on the definition of "mean proportional" and "third proportional," when applied to negative numbers. There is no need on our part to enter into a minute discussion of the validity of the arguments presented. Bernoulli's argument involving infinite areas between the hyperbola and its asymptotes was repeatedly bombarded during the eighteenth century, but was never hit at its vulnerable point, namely the assumption

[1] *Ex quibus intelligitur, in ipsis rei Analyticæ fundamentis aliqua adhuc neglecta fuisse.*

that $\infty - \infty = 0$.[1] It is interesting to note that logarithms are part of the time connected with the two progressions, as in Napier's definition of a logarithm, but most of the time with the exponential concept as expressed in the exponential notation of the present time. Leibniz makes few appeals to geometry; J. Bernoulli uses curves repeatedly, as if more could be gotten out of a figure than is put into it.

Leibniz insists that $\log(-1)$ and $\log(\sqrt{-1}n)$ do not exist. *Non potest dari Logarithmus* $\sqrt{-2}$. Non-existence is based on inconceivability. We shall see later that Euler puts a different intepretation upon Leibniz; he represents Leibniz as contending that $\log(-1)$ is imaginary, not as non-existing. Leibniz died three years after the close of this controversy. This correspondence between him and John Bernoulli during the years 1712 and 1713 was not published until 1745. Not until then did the logarithms of negative numbers engage the attention of mathematicians in general.

Meanwhile there are other researches demanding our attention. In an article in the *Philosophical Transactions* of London, published in 1714, Roger Cotes develops an important formula which in modern notation is

$$i\varphi = \log(\cos\varphi + i\sin\varphi).^2$$

In 1722, after the death of Cotes, this article was republished in his *Harmonia mensurarum*. He introduces as the "measure of a ratio" (mensura rationis) the logarithm of the ratio, multiplied by a constant or modulus. As Braunmühl points out, this "measure of a ratio" was long lost sight of, but was introduced anew in the nineteenth century. We have already called attention to the fact that Cotes himself was anticipated by Edmund Halley in this mode of measuring ratio.

Without stopping to explain how "the measure of the ratio" figures in Cotes's

[1] Proofs involving the comparison with each other of infinite areas in a plane appeal to our intuition with great force. They were accepted as valid by some writers of the nineteenth century as well as of the eighteenth century. Louis Bertrand based upon such comparison his proof of Euclid's parallel-postulate. See L. Bertrand, *Developpement nouveau de la partie elementaire des mathematiques*, T. II, a Geneve, 1778, pp. 19, 20. This proof was accepted by Johann Schultz. See J. Schultz, *Versuch einer genauen Theorie des Unendlichen*, Königsberg und Leipzig, 1788, pp. xi–xv. Similar reasoning was endorsed by Johann Heinrich Lambert. See Lambert, *Deutscher gelehrter Briefwechsel*, Bd. I, Berlin, 1781, p. 118, in a letter dated Feb. 2, 1766. Substantially Louis Bertrand's proof was published anonymously in Crelle's *Journal* in 1834. It was accepted as valid by A. De Morgan. See De Morgan on "Infinity and the Sign of Equality" in *Trans. Cambridge Phil. Society*, Vol. XI, Part I, p. 158. It was translated and published by W. W. Johnson in the *Analyst* (Des Moines, Vol. III, 1876, p. 103). See Cajori, *Teach. u. Hist. of Math. in U. S.*, 1890, p. 379.

[2] That Cotes derived this relation was pointed out by Timtschenko in his History of the Theory of Functions, 1899 [Russian], pp. 519–522. It receives emphasis also in an article by A. v. Braunmühl in *Bibliotheca mathematica*, 3d S., Vol. V, 1904, pp. 355–365. How it happened that so important a theorem should remain in the writings of Cotes for 185 years, without being detected, may perhaps be inferred from Cotes' mode of statement: "Nam sit quadrantis circuli quilibet arcus, radio CE descriptus, sinum habeat CX sinumque complementi ad quadrantem XE; sumendo radium CE pro Modulo, arcus erit rationis inter $EX + XC\sqrt{-1}$ et CE mensura ducta in $\sqrt{-1}$."

derivation of his formula for $i\varphi$, his process may be roughly outlined in modern notation as follows: The area of the surface generated by an arc of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, when revolved around the Y-axis and taken between the limits $y = 0$, $y = y$, can be expressed in two ways, namely,

$$S = a\pi\left\{y\sqrt{1+\frac{a^2-b^2}{b^4}y^2} + \frac{b^2}{\sqrt{a^2-b^2}}\log\left(y\sqrt{\frac{a^2-b^2}{b^4}} + \sqrt{1+\frac{a^2-b^2}{b^4}y^2}\right)\right\},$$

$$S = a\pi\left\{y\sqrt{1-\frac{b^2-a^2}{b^4}y^2} + \frac{b^2}{\sqrt{b^2-a^2}}\varphi\right\},$$

where $\sin\varphi = y\sqrt{\frac{b^2-a^2}{b^4}} = iy\sqrt{\frac{a^2-b^2}{b^4}}$, $\cos\varphi = \sqrt{1+\frac{a^2-b^2}{b^4}y^2}$. Comparing the two expressions for S, we have

$$i\psi = \log(i\sin\psi + \cos\psi). \tag{1}$$

While Cotes was anticipated by John Bernoulli I in establishing a relation between the logarithm of an imaginary number and a goniometric function, Cotes, in turn, anticipated the continental mathematicians in the derivation of $i\varphi = \log(\cos\varphi + i\sin\varphi)$. It was much later, in a letter of Oct. 18, 1740, that Euler stated to John Bernoulli I that $y = 2\cos x$ and $y = e^{+x\sqrt{-1}} + e^{-x\sqrt{-1}}$ were both integrals of the differential equation $d^2y/dx^2 + y = 0$; that the two integrals were equal to each other, since both could be expanded into the same infinite series. Euler makes remarks from which it follows that he knew at that time the corresponding exponential expression for *sin x*.[1] Both expressions are given by him in *Miscellanea Berolinensia* 1743 and again in his *Introductio in analysin*, Lausannæ, 1748, Vol. I, p. 104, where he gives also the all-important formulæ, $e^{+v\sqrt{-1}} = \cos v + \sqrt{-1}\sin v$, $e^{-v\sqrt{-1}} = \cos v - \sqrt{-1}\sin v$. From Cotes's formula $i\varphi = \log(\cos\psi + i\sin\psi)$, given in 1714 and 1722, to Euler's exponential form is an easy step, yet over a third of a century intervened between the first publication of the one and of the other. It is interesting to observe that neither Cotes nor Euler appear to hesitate in, or to recoil from, the use of $\log(\cos\psi + i\sin\psi)$, involving the logarithm of complex numbers. Moreover, neither Cotes, nor Euler in his *Introductio*, make any attempt to use this relation in the discussion of the theory of logarithms of complex numbers. Both were aware of the periodicity of the trigonometric functions. Had Cotes applied the idea of periodicity to $i\varphi = \log(\cos\psi + i\sin\psi)$ he might have anticipated Euler by many years in showing that the logarithm of a number has an infinite number of different values.

The theory of logarithms of negative numbers was incidentally touched by Euler very early, in his correspondence with John Bernoulli I. The letters which passed between these men in 1727–1731 have been in the possession of the Stockholm academy of sciences and have for the first time been published in full by

[1] *Bibliotheca mathematica*, 2d S., Vol. 11, 1897, pp. 48–49.

G. Enestrom in 1902.[1] A translation from the original Latin into German was brought out by E. Lampe.[2] It will be remembered that Euler was a pupil of John Bernoulli I and had followed the two sons (Daniel and Nicolaus) of John Bernoulli to St. Petersburg. Euler was then 20 years old; John Bernoulli was 60. The following is a synopsis of the correspondence, in so far as it bears on logarithms:

Nov. 5, 1727. Euler to J. Bernoulli: The equation $y = (-1)^x$ is difficult to plot, since y is now positive, now negative, now imaginary. It cannot represent a continuous line.[3]

Jan. 9, 1728. J. Bernoulli to Euler: If $y = (-n)^z$, then $ly = zl(-n)$ and $dy/y = dx \cdot l(-n) = dx \cdot l(+n)$; for $dl(-z) = -dz/-z = +dz/+z = dlz$. Integrating, $ly = xln$, and $y = n^x$. Hence $y = (\pm 1)^x$ becomes $1^x = 1$, or $y = 1$.

Dec. 10, 1728. Euler to J. Bernoulli: I have arguments both for and against $lx = l(-x)$. If $lxx = z$, we have $\frac{1}{2}z = l\sqrt{xx}$. But $\sqrt{xx}$ is as much $-x$ as $+x$. Hence $\frac{1}{2}z = lx = l(-x)$. It may be objected that xx has two logarithms, but whoever claims two, ought to claim an infinite number.[4] Argument against: From the equality of the differentials we cannot infer the equality of the integrals. Moreover, $l(-x) = lx + l(-1)$; hence $l(-x) = lx$ only if $l(-1) = 0$. Again, if $lx = l(-x)$, then $x = -x$ and $\sqrt{-1} = 1$, but I rather think the conclusion from the equality of the logarithms to the equality of the numbers cannot be drawn. Your expression for the area of a circular sector[5] of radius a, viz. $aa/(4\sqrt{-1}) \times l(x+y\sqrt{-1})/(x-y\sqrt{-1})$, becomes for a quadrant, x being then 0, $aa/(4\sqrt{-1}) \cdot l(-1)$. Hence, if $l(-1) = 0$, we must have $\sqrt{-1} = 0$ and even $1 = 0$. Most celebrated Sir, what do you think of these contradictions?

April 18, 1729. J. Bernoulli to Euler: When I say that $lx = l - x$, it is to be understood that $l - (x)$ is meant, not $l(-x)$. Thus, $l - (x)^{\frac{1}{2}}$ is real, but $l(-x^{\frac{1}{2}})$ is imaginary. The area of the circular sector is 0, when $x = 0$, however much it ought to be equal to the quadrant. Let the constant Q be a quadrant, then we may write the area of the sector generally $= aa/(4\sqrt{-1})l(x+y\sqrt{-1})/(x-y\sqrt{-1}) + nQ$, so that, the first term vanishing when the sector becomes a quadrant, n may be so chosen as to make nQ any multiple or submultiple of the quadrant we need. For a semi-quadrant we have $aa/(4\sqrt{-1})l\sqrt{-1}$, which is 0, since $l\sqrt{-1} = 0$. Here we must take $n = \frac{1}{2}$.

May 16, 1729. Euler to J. Bernoulli: The difference between $l - (x)$ and $l(-x)$ is not clear to me. The expression $4a/(4\sqrt{-1})l(x+y\sqrt{-1})/(x-y\sqrt{-1})$, thought to be constant, appears to me to be increasing, since $x = 0$ exhibits a vanishing sector. That nQ ought to be added, I do not as yet see. If n can be $\frac{1}{2}$, it can be $\frac{1}{4}$ and any number. It would be superfluous to show that $aa/(4\sqrt{-1})l(x+y\sqrt{-1})/(x-y\sqrt{-1})$ must express a sector, if nQ alone were sufficient to represent any sector whatever. Neither of us can afford to run into paralogisms.

In this correspondence between John Bernoulli I and L. Euler, Bernoulli holds the view that $\log n = \log(-n)$. This is the same formula which he defended 16 years earlier, in letters to Leibniz. Now, as then, Bernoulli argues from the equality of two differentials to the equality of the two resulting general

[1] See *Bibliotheca mathematica*, 3d S., Vol. 4, 1902, pp. 344–388. Information on these letters is given also in the same journal, 2d S., Vol. 11, 1897, pp. 51–56; Vol. 13, 1899, p. 46.

[2] *Festschr.* z. Feier d. 200. Geburtstages L. Eulers, Leipzig and Berlin, 1907, pp. 119–137.

[3] In this synopsis we follow the notation used in the letters except in our use of slanting fractional lines and parentheses necessitated by them.

[4] *Posset quidem objici, xx habere duos logarithmos, sed hoc qui asser[ere vult] infinitos adjudicare deberet.*

[5] To facilitate the derivation of this expression, we note that

$$\int (a^2 - x^2)^{-\frac{1}{2}}\,dx = -i\int (x^2 - a^2)^{-\frac{1}{2}}\,dx = -i\log(x + \sqrt{x^2 - a^2}) + c$$

$$= -i/2\log(x+iy)^2/(x^2+y^2) - i/2\log a^2 + c = -i/2\log(x+iy)/(x-iy) - i/2\log a^2 + c.$$

integrals. In his argument on sectorial areas (Apr. 18, 1729) he confuses definite integrals with general integrals. His distinction between $l-(x)^{\frac{1}{2}}$ and $l(-x^{\frac{1}{2}})$ is not made clear. Perhaps he means simply that $l(-x^{\frac{1}{2}})$ shall signify $l(-x)^{\frac{1}{2}}$, when x itself is positive.

Euler's argument, that $e^{z/2}=\pm x$ yields $\log x=\log(-x)$, involves an interesting point. When we write $a^b=c$ and define $b=\log_a c$, a and c are taken to be both single-valued. Euler drops this restriction on c. He takes $e^{z/2}=\pm x$, so that the definition implied in his mode of procedure, viz. $z/2=\log(\pm x)$, really amounts to two definitions, $z/2=\log x$ and $z/2=\log(-x)$. Now there is no objection, *a priori*, to two distinct definitions. In vector analysis we have at least two definitions of multiplication, yielding a vector-product $a\times b$, and a scalar product $a\cdot b$. The question to be considered is, can the two definitions be used side by side? Do the two fit together so as to give a non-contradictory logarithmic theory of complex numbers? The conclusion is drawn from the two definitions and the ordinary rules of operation, that all roots of $+1$ and -1, in other words, all complex numbers of unit modulus, have zero for their logarithm. This is certainly very simple, also quite useless. Previous to the Euler-Bernoulli correspondence no real contradictions had been pointed out in this theory. It was Euler who gave it a death blow by pointing out that $\log\sqrt{-1}=0$ was in conflict with the formula $\sqrt{-1}\,\pi=2l\sqrt{-1}$, resulting from J. Bernoulli's accepted expression for the area of a circular sector. The blow dealt by Euler was not considered fatal at the time. Definite integrals were as yet indefinitely comprehended. J. Bernoulli's theory of logarithms continued to find defenders. Euler made another important remark in his letter of December 10, 1728; he touches for the first time the truth that $\log n$ has an infinite number of values. But he does not pursue this matter further at this time.

THE UNION OF THE LOGARITHMIC AND EXPONENTIAL CONCEPTS.

The possibility of defining logarithms as exponents was recognized in the seventeenth century by John Wallis, but not until about 1742 do we find a systematic exposition of logarithms based on this idea. About this time it came to be recognized that involution has two inverses, different in kind, namely, evolution and logarithmation; in the first inverse we assume, in $a^b=c$, b and c as given and find a, in the second inverse we assume a and c as given and find b. Tropfke[1] names William Gardiner as the first to give the new definition of logarithms and to base the theory of logarithms upon it. It is given in the introduction to Gardiner's *Tables of Logarithms*, London, 1742. The definition is as follows: "The common logarithm of a number is the Index of that power of 10 which is equal to the number." It is practically certain that this definition is not due to Gardiner, but to William Jones. "The Explication of the Tables," says Gardiner, ". . . I have collected wholly from the papers of W. Jones, Esq." Maseres,[2] who reprinted this "Explication" in 1791, attributed it entirely to

[1] Tropfke, *op. cit.*, Vol. II, 1903, p. 142, note 576.
[2] Maseres, *Scriptores logarithmici*, Vol. II, London, 1791, p. 1.

Jones. Prof. W. W. Beman informs me that Jones's exposition of logarithms, as given in his *Synopsis palmariorum matheseos*, 1706, was based on Halley's treatment of 1695, but a posthumous paper by Jones, published in the *Philosophical Transactions* for the year 1771, gives the new definition. Whether Jones printed this definition before Gardiner is still undetermined. The one whose influence was greatest in emphasizing the new view was Euler, who, in his *Introductio*, 1748, Chap. VI, §102, gives the definition involving exponents. In this same chapter Euler gives an exposition of negative and fractional exponents and calls attention to the multiple values of a number having a fractional exponent, an explanation seldom found in mathematical treatises of that time. That the new definition of a logarithm was in every way a step in advance has been doubted by some writers. Certain it is that it involves internal difficulties of a serious nature.

[*To be continued.*]

MINIMUM COURSES IN ENGINEERING MATHEMATICS.

By S. EPSTEEN, University of Colorado.

Introduction.—This paper was suggested by a number of inquiries as to the nature and content of the course in engineering mathematics at the University of Colorado. This course is based on three entrance units in Mathematics, and consists of algebra, trigonometry, analytic geometry, the calculus, and least squares as prescribed courses and differential equations, higher calculus, vector analysis, Fourier's series and other advanced courses as electives.

The following outline[1] is not that of a complete course in engineering mathematics, nor even the average course, but, as the title of the paper indicates, the minimum course. The average course is based on this minimum but contains more material, both theoretical and applied. This outline gives an irreducible minimum. A course falling below this standard may be a good trade school course, it may be a most useful and practical course in many respects, but it is not a course in engineering mathematics.

Mathematics and Engineering.—Engineering mathematics is in no sense trade school mathematics or practical arithmetic. A trade school may have little use for mathematics as a science, but the engineering college demands a knowledge of principles as well as facts. This is particularly noticeable in the recent advances in the profession of civil engineering which have been along the lines laid down by Rankine and not by Trautwine. To the engineering student mathematics is as essential as anatomy is to the surgeon, as chemistry is to the apothecary, as drill is to the army officer. The professor of engineering is certainly on firm ground when he takes the stand that the mathematics taught to his students should not be too abstract on the one hand nor too concrete on the other. If the subject matter is too abstract it is unintelligible or uninteresting to the beginner; if it is

[1] The Editors, while in sympathy with the broad purposes of this paper, share no responsibility for the details of the suggested programs.

too concrete the science degenerates to the mere performing of certain mechanical operations—to a common tool instead of a valuable instrument.

Mathematicians and Engineers.—Until very recently there was a complete lack of understanding and sympathy between mathematicians and engineers. In the proceedings of the Society for the Promotion of Engineering Education, volume 17, page 39, the chairman of the joint committee makes the following statement: "The mathematicians expected that the engineers would tell them that what they wanted is practical arithmetic, while the engineers fully expected that the mathematicians would tell them that what they wanted is 'pure mathematics,' the queen of the sciences.' But as the session progressed everyone was astonished to find that no conflict was precipitated." The situation has greatly improved since that time. In volume 19 of the same proceedings we find a complete syllabus of a minimum course in mathematics for engineering students. The report is by an individual, not by the entire committee. While the author's eminence as a mathematician is above all question, his experience with engineering is not as extensive as his experience in pure mathematics. As might be expected under such circumstances, his report is substantially sound, but it needs revision in details. For instance, in the algebra course, no mention is made of graphic algebra. This topic should be included. On the other hand, harmonic progressions are mentioned, with the comment that they are not of great importance. This topic should be excluded.

The Engineering Student and the Arts Student.—The purpose and aim of the prospective engineer are so radically different from that of the general student that the content of the course should be different from that which is generally given to the student who elects mathematics as a part of a general education. Experience proves that the best instruction to engineering students is not given by instructors who have no patience with anything except pure mathematics, but by those who have had some training in engineering or at least considerable training in mathematical physics. That is to say, the most sympathetic instructors are generally those who are near enough to engineers to take some interest in the concrete problems themselves as distinct from their solutions.

The Freshman and Sophomore Years.

Entrance Requirements.—Many high schools are rebelling against college domination. Their stand is well taken. Only a small percentage of their students go to college and it is therefore the obvious duty of the high school to concentrate its chief efforts on the majority. Most engineering colleges demand three units in mathematics for entrance, consisting of one and one-half years of algebra, one year of plane geometry and one-half year of solid geometry. Under the existing circumstances this is all that can be effectively enforced, at least in some parts of the country, even if more is desirable, and even though more is frequently offered.

Engineering Algebra.—While the amount of time devoted to each course differs considerably in the various colleges, the general average for algebra is

three times per week for one semester (or the equivalent); that is, 54 class periods less holidays and quiz days, making a total of 45 class periods. In view of the fact that the course must necessarily begin with a brief review, it is perfectly obvious that the ordinary text book contains far too much material. Many authors argue that this is desirable in that it gives the instructor a wider range of selection and it impresses upon the student the fact that when he has finished the course he has not learned all there is to know about it. Whatever merit or demerit there may be in the second argument, the first is not strictly sound. Even an experienced instructor does not always know what to select from the immense amount of material at his disposal, while the inexperienced instructor is quite at sea. By inexperienced is meant not merely those who have done little or no teaching, but also those whose interests are exclusively in pure mathematics and who have had no practical experience with engineers, their points of view and their standards of value. For example, even experienced instructors frequently select theory of equations as a suitable topic, whereas, in reality, this chapter is relatively unimportant to an engineer as compared with topics that are omitted when this one is included. An actual inquiry shows that the professors of electrical engineering and mechanical engineering place almost no value on this topic; and the professors of civil engineering state that while they occasionally want the students to know how to solve a cubic or a quartic equation, the need is so rare that they would willingly forego this topic for others.

The minimum course in algebra should consist of the following chapters: (1) Review of fractional and negative exponents, reduction of surds, imaginaries, zero, infinity; (2) Logarithms; (3) Variables, functions, graphs; (4) Simultaneous linear equations, determinants of the second and third orders, graphical solution; (5) Quadratic equations; (6) Simultaneous quadratics; (7) Variation; (8) Arithmetic and geometric progressions (number of terms finite, infinite); (9) Binomial theorem, the series for e and e^x; (10) Complex numbers, vectors, addition, subtraction, multiplication and division; (11) Partial fractions; (12) Permutations, combinations, probability. There are about forty formulas in these chapters which should be understood and memorized so that they may become the student's everyday working tools—his mental machinery. There should be numerous applications to simple problems taken from actual engineering practice.

Engineering Trigonometry.—In trigonometry the problem of a suitable course is much simpler than in algebra. The general average of time allowed to this course is two hours per week for one semester or the equivalent; that is, 36 days less holidays and quiz days, making about 30 class periods. Trigonometry is needed primarily for three specific purposes—surveying, physics, calculus. The minimum course can therefore be summarized in about forty formulas under the following headings: (1) Trigonometric functions; (2) Right triangles; (3) The addition theorems; (4) Inverse trigonometric functions, trigonometric equations; (5) The oblique triangle; (6) Circular measure, graphical representation. If the students have had logarithms early in the algebra course, so that this topic may be omitted in the trigonometry, it is possible to

get in a short chapter on the right spherical triangle. The students of civil engineering need spherical trigonometry in some of their other courses. A brief study of the right spherical triangle is not enough, but it constitutes an introduction, and this is all that can be reasonably expected under the time limitations.

Engineering Analytics.—Most colleges devote five hours per week for one semester to analytic geometry; that is, about 80 actual class periods. While one cannot cover this subject fully in that time, it is sufficient for a fair introduction. Indeed, if eighty periods could be given to algebra also, instead of forty-five, the results would be large in proportion to the extra time. Analytic geometry is an indispensable part of any good mathematical curriculum. Although highly interesting in itself, it would be a still better course if frequent applications were introduced. The newer books on algebra, trigonometry and calculus abound in interesting and important applications, but not so in analytic geometry. Authors of the text books on this subject still follow slavishly in the footsteps of their predecessors and concrete applications are exceedingly rare in a book on analytics.

The minimum course should consist of about seventy-five formulas under the following headings: (1) The straight line; (2) The circle; (3) The parabola; (4) The ellipse; (5) The hyperbola; (6) Polar coordinates; (7) Transformation of coordinates; (8) Higher plane curves; (9) Coordinates in space. It is well to take up a chapter on the general equation of the second degree if time permits, but this is by no means an essential part of the minimum course, as the author of the syllabus (Art. 5) thinks. Indeed, a chapter on higher plane curves, including the cycloid, hypocycloid, epicycloid, involute and others, is far more important.

Engineering Calculus.—Calculus is doubtless the most valuable and effective instrument the engineer can possess. Of all the subjects in the curriculum no other is so far reaching and powerful in its applications. Still, one sometimes hears a successful engineer say "I use algebra and trigonometry a great deal but never the calculus." Whenever a man makes this assertion, it usually indicates that he had defective instruction in the calculus when he first took the course. Naturally, he could not use an instrument which he did not understand. He may have been a good engineer without the calculus, but he would have been a better engineer with it.

A minimum course in the calculus should consist of a thorough understanding of the proofs and applications of about one hundred formulas and theorems included in the following chapters: (1) Functions, continuity, limits (a very brief chapter); (2) Differentiation; (3) Simple applications to geometry, physics, mechanics, maxima and minima; (4) Integration as anti-differentiation; (5) Integration as the limit of a sum, definite integrals; (6) Applications of integration; (7) Centers of mass, moments of inertia, radii of gyration (special emphasis on plane sections); (8) Series, indeterminate forms; (9) Approximate methods of integration; (10) Differential equations with applications. All the important formulas and theorems should be memorized.

The Junior Year.

Least Squares.—The subject of least squares should be required of all students in civil engineering, and probably of those in mining engineering, in the first half of the junior year. Differential equations should be offered as an elective in the second half year.

The minimum course in least squares should consist of about forty formulas under the following headings: (1) Review of theory of probability; (2) Errors and their probability, probability curve, probability integral; (3) Important processes such as solution of observational equations (equal and unequal weights), average error, percentage error, mean error, approximation formulas, propagation of errors; (4) Applications to civil engineering and physics; (5) Normal equations; (6) Fitting formulas to observations.

Many mathematicians intensely dislike least squares and do not want to teach the course. The sentiment seems to be founded on the fact that the ideas and methods of least squares are foreign to a mathematician's usual mode of thought and standards of judgment. The subject does not belong to the domain of logical mathematics, and it is thus different from the usual course in pure mathematics. It belongs to the domain of experimental mathematics, and when viewed in the right light, is really fascinating.

Differential Equations.—The subject of differential equations should be required of all students in electrical, chemical and mechanical engineering in the second half of the junior year. Least squares should be offered as an elective in the first half of the year. In a three hour course the following ground can be covered: (1) Formation of differential equations; (2) Equations of first order and first degree; (3) Equations of first order but not of first degree; (4) Singular solutions; (5) Applications to physics, mechanics, geometry; (6) Linear equations of second order with constant coefficients, with applications; (7) Linear equations with variable coefficients, with applications; (8) Equations of particular forms; (9) Miscellaneous applications.

The Senior Year.

Advanced Courses.—The courses for seniors and graduates should be elective. Almost any advanced course may be offered, although it is well to emphasize those which have some connection with applications. Some of the most important courses are: (1) Determinants and theory of equations; (2) Advanced calculus; (3) Fourier's series; (4) Bessel's functions; (5) Spherical and cylindrical harmonics; (6) Vector analysis; (7) Elliptic functions; (8) Theory of functions; (9) Partial differential equations; (10) Linear differential equations.

Advanced Calculus.—Very few departments of mathematics are large enough to offer all the foregoing courses each year. The best that can be expected is to have one or two each semester. Advanced calculus is, in a sense, one of the most important in the list and is worthy of special mention. The minimum course for engineers should include short chapters of an introductory nature on: (1) Infinite series; (2) Definite integrals; (3) Hyperbolic functions: (4) Elliptic functions, Bessel's functions; (5) Beta and gamma functions; (6) Fourier's series.

Conclusion.—Each branch of engineering demands a somewhat different mathematical training from every other branch. For this reason every text book and every course must necessarily be a compromise. There are on the market some very acceptable books on algebra, trigonometry and the calculus. The pressing need is for equally acceptable books on analytic geometry, least squares and differential equations. There are, indeed, some very good American texts on differential equations, but, with possibly one exception, it does not appear that their authors have made any particular effort to find out which chapters are most useful to engineers, or to provide applications adapted to the needs of engineers, beyond certain standard types well known to all mathematicians.

A GEOMETRIC INTERPRETATION OF THE FUNCTION *F* IN HYPERBOLIC ORBITS, CORRESPONDING TO THAT OF THE ECCENTRIC ANOMALY *E* IN ELLIPTIC ORBITS.[1]

By W. O. BEAL, Illinois College.

The Geometric Interpretations of *E* and *M* in Elliptic Orbits.

If the elements of an elliptic orbit of a planet are given in the problem of the relative motion of two bodies, subject only to their mutual gravitation, the polar coördinates $r = SP$ (Fig. 1) and $v = \angle ASP$, the radius vector and true anomaly of the planet, can be computed for any time t in the following manner:

Compute the mean daily motion

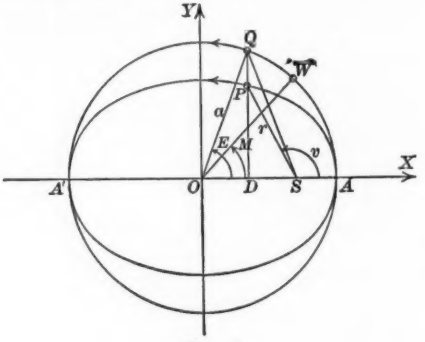

Fig. 1.

$$n = \frac{k\sqrt{m_1 + m_2}}{a^{\frac{3}{2}}},$$

[1] This problem is given by F. R. Moulton in his *Introduction to Celestial Mechanics*, p. 159.

the mean anomaly $M = n(t - T)$, the eccentric anomaly E, by solving Kepler's equation

$$E - e \sin E = M,$$

and then $r = a(1 - e \cos E)$, and

$$\tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}.$$

In these, a, T and e stand for the semi-major axis, time of perihelion passage and eccentricity of the relative orbit. The function E is first introduced into the problem analytically to aid in the integration of the equation of areas. It has the well known geometric meaning that it is always equal to the central angle AOQ, in the auxiliary circle, which corresponds to the true anomaly v.

We may give a geometric meaning to M by imagining a point W moving with uniform speed along the auxiliary circle in such a way that it would be at the perihelion point A of the orbit every time the planet was there. Thus M would be the central angle AOW described by the radius vector of this point.

It is further true that the area of the

$$\text{sector } AOQ = \tfrac{1}{2}a^2E,$$

and

$$\text{triangle } OSQ = \tfrac{1}{2}a \cdot ae \sin E.$$

Hence

$$\text{Area } ASQ = \tfrac{1}{2}a^2(E - e \sin E) = \tfrac{1}{2}a^2M$$

$$= \text{Area sector } AOW.$$

Thus M and E are proportional to the areas generated by SQ and OQ respectively, and would be numerically equal to twice those areas if $a = 1$.

THE GEOMETRIC INTERPRETATIONS OF F AND M IN HYPERBOLIC ORBITS.

The radius vector and true anomaly, r and v, in a hyperbolic orbit can be found in like manner for any instant t by the formulas,

$$(1) \qquad \begin{cases} n = \dfrac{k\sqrt{m_1 + m_2}}{a^{\frac{3}{2}}}, \\ M = n(t - T), \\ -F + e \sinh F = M, \\ r = a(-1 + e \cosh F), \\ \tan \dfrac{v}{2} = \sqrt{\dfrac{e+1}{e-1}} \tanh \dfrac{F}{2}. \end{cases}$$

Like E, the function F is first introduced analytically as an auxiliary quantity.

We are to find geometric meanings for F and M analogous to those given to E and M in an elliptic orbit. For this purpose use the rectangular hyperbola having the same center and semi-major axis a as the orbit, and call it the auxiliary hyperbola. Also use the auxiliary circle with radius a and center at the center of the orbit. Thus in Fig. 2, $P'AP$ represents a hyperbolic orbit, with focus at S, perihelion at A, center at O, and P any position on the orbit; also $Q'AQ$ is the auxiliary hyperbola, with Q corresponding to P.

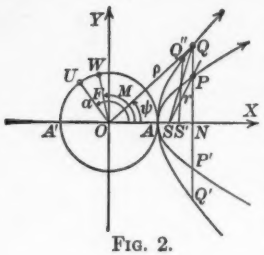

FIG. 2.

Join Q to O, and call $\angle NOQ = \psi$, $OQ = \rho$. Then the equation of the auxiliary hyperbola in rectangular coördinates is

$$x^2 - y^2 = a^2, \tag{2}$$

which expressed in polar coördinates gives $\rho^2(\cos^2\psi - \sin^2\psi) = a^2$, or

$$\rho^2 = \frac{a^2}{\cos 2\psi}. \tag{3}$$

Next compute the area bounded by OA, OQ and arc AQ.

$$OAQ = \tfrac{1}{2}\int_0^{\psi} \rho^2 d\psi = \frac{a^2}{2}\int_0^{\psi} \sec 2\psi d\psi = \frac{a^2}{4}\log\tan\left(\frac{\pi}{4}+\psi\right) = \frac{a^2}{2}K, \tag{4}$$

where we place

$$K = \tfrac{1}{2}\log\tan\left(\frac{\pi}{4}+\psi\right). \tag{5}$$

From this we readily deduce

$$\tan\psi = \tanh K. \tag{6}$$

Now the rectangular coördinates of Q may be expressed in terms of a parameter G by the relations

$$\begin{cases} x = ON = a\cosh G, \\ y = NQ = a\sinh G, \end{cases} \tag{7}$$

for these expressions satisfy (2). But

$$\tan\psi = \frac{NQ}{ON} = \frac{a\sinh G}{a\cosh G} = \tanh G. \tag{8}$$

Hence by (6)

$$K = G. \tag{9}$$

Again the area of the triangle

$$OSQ = \tfrac{1}{2}OS \cdot NQ = \tfrac{1}{2}ae \cdot a\sinh G. \tag{10}$$

Hence the area

$$ASQ = \tfrac{1}{2}a^2(e\sinh G - G). \tag{11}$$

Remembering that the changes in G are due to the motion of P, we have by comparison of (11) with (1)

$$G = F, \tag{12}$$

and

$$ASQ = \frac{a^2}{2}M; \tag{13}$$

also (4) becomes

$$OAQ = \frac{a^2}{2}F. \tag{14}$$

Thus M and F are proportional to the areas generated by SQ and OQ respectively, and would be numerically equal to twice those areas if $a = 1$.

But $M = n(t - T)$, that is, M is proportional to the time, from which it follows that as P describes its hyperbolic orbit, the corresponding point Q moves on the auxiliary hyperbola so that the law of areas holds with respect to the focus of the orbit.

From the form of the expressions (13) and (14) which represent the areas ASQ and OAQ, it is apparent that those areas are equivalent to circular sectors in a circle with radius a and central angles M and F respectively. We can thus imagine two points W and U traveling on the circle ARA', so related to the planet P that when P passes the perihelion point A, they will also pass through A. One of them, W, will go around the circle at a uniform speed, with its radius vector describing the area $\frac{a^2}{2}M$ = area ASQ. The other, U, will move around the circle at a non-uniform speed, with its radius vector OU describing the area $\frac{a^2}{2}F$ = area OAQ.

Now from (1)

$$\begin{cases} \dfrac{dM}{dt} = \dfrac{k\sqrt{m_1 + m_2}}{a^{\frac{3}{2}}}, \\ \dfrac{dF}{dt} = \dfrac{k\sqrt{m_1 + m_2}}{a^{\frac{3}{2}}(e \cdot \cosh F - 1)}, \end{cases} \tag{15}$$

which show that for $e < 2$ the non-uniformly moving point U would be moving faster than W at A; that they would be moving with equal speeds when

$$\cosh F = \frac{2}{e},$$

and thereafter the uniformly moving point W would move the faster.

By placing $F = M$ in (1) we find that W would overtake U at the time determined by

$$\sinh M = \frac{2M}{e}.$$

(15) also shows that the speed of the non-uniformly moving point would always decrease but not be zero. From (6) it follows, since $K = F$, that in an indefinitely great period of time this point U, as well as W, would make an indefinitely great number of revolutions about the circle.

The law of areas for hyperbolic motion gives the sector $ASP = \frac{a^2}{2}\sqrt{e^2 - 1}\, M$.

For a rectangular hyperbola $e = \sqrt{2}$ which substituted in this gives $\frac{a^2}{2} M$, or (13).

Hence it follows that the area described by SQ is equal to the area described in an equal time by the radius vector $S'Q''$ of a planet Q'' having the same mass and major axis as the planet P, but moving in an orbit which is a rectangular hyperbola.

THIRD CLEVELAND MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

As many readers of the MONTHLY may be unfamiliar with the history and the aims of the American Association for the Advancement of Science, the following data relating to these subjects may be of interest. In 1847 the Association of American Geologists and Naturalists agreed to resolve itself into the American Association for the Advancement of Science, and decided to hold the first meeting under the new management at Philadelphia in 1848.

At this first meeting of the Association its objects were stated to be as follows: "The objects of the Association are, by periodic and migratory meetings, to promote intercourse between those who are cultivating science in different parts of the United States, to give a stronger and more general impulse, and a more systematic direction to scientific research in our country, and to procure for the labors of scientific men increased facilities and a wider usefulness."

These objects have been modified only slightly during the sixty-four years since the Association was formed. For instance, the term United States has been replaced by the more general term America, and several of the meetings of the Association have been held in Canada. While most of the meetings have been in the east, the migrations have extended as far west as Denver and as far south as New Orleans. Four meetings have been held in each of the two cities Buffalo and Washington; and three in each of the four cities, Boston, New York, Philadelphia, and Cleveland. A few other cities have entertained the Association twice, but in most cases only a single visit to a given city has been made.

Similar organizations exist in many other countries. The British Association, organized in 1831, seems to occupy an especially prominent place in the scientific activity of that country and in the extent of its migrations. Its recent visits to South Africa and to Winnipeg, Canada, were marked with great liberality on

the part of the countries visited; and, according to *Nature*, the Commonwealth of Australia has already granted £15,000 to be handed to the British Association to cover the passages of not fewer than 150 British and foreign men of science, who will attend the meeting of the British Association in Australia in 1914.

While the American Association has thus far confined its migrations to much narrower limits than its prototype, it has already had under consideration meetings on the Pacific Coast as well as in the Hawaiian Islands, and with the growth of scientific activity, it also will doubtless further extend its sphere of influence, by direct contact, to other parts of the country. It has now a membership of more than 8,000 and most of the leaders of American science have been found on its register since its organization.

The scientific work of the Association is conducted by its eleven sections and by the affiliated national societies. About twenty-five of the latter met at Cleveland and their programs formed the most important feature of this meeting. Those which are of especial interest to mathematicians are as follows: The American Mathematical Society, including the Chicago Section; The Astronomical and Astrophysical Society of America; The American Physical Society; The American Federation of Teachers of the Mathematical and the Natural Sciences.

Section A of the Association is devoted to Mathematics and Astronomy. "As it is the policy of the Association to avoid competition with programs presented before special national societies," and as both of the subjects included in the scope of Section A were represented by national societies, this Section arranged for only one session, which was held on Tuesday afternoon, December 31, 1912. During this session the addresses of the retiring vice-presidents of Sections A and B were presented.

In the absence of the retiring vice-president and chairman of Section A, Professor E. B. FROST, Director of Yerkes Observatory, his address, entitled "The spectroscopic determination of stellar velocities, considered practically," was read by Professor J. A. PARKHURST. This address will be published in *Popular Astronomy*. The retiring vice-president and chairman of Section B Professor R. A. MILLIKAN, University of Chicago, presented an address entitled "On unitary theories in physics," which has appeared in *Science*.

In addition to these two addresses the following four papers were presented during the given session of Section A:

(1) Henri Poincaré as a mathematical physicist. By Professor A. G. WEBSTER, Clark University.

(2) Some general aspects of modern geometry. By Professor E. J. WILCZYNSKI, University of Chicago.

(3) Cosmical magnetic fields. By Dr. L. A. BAUER, Director of the Department of Terrestrial Magnetism, Carnegie Institution of Washington.

(4) Preliminary note on an attempt to detect the general magnetic field of the sun. By G. E. HALE, Director of Mt. Wilson Observatory.

In the absence of Professor Hale his paper was presented by Dr. BAUER.

The others were presented by their respective authors. The programs of the affiliated societies were, in general, very much more extensive than those of the Sections. For instance, the program of the American Mathematical Society contained fifty-four titles and the American Physical Society had an almost equally extensive list.

While such meetings afford excellent apportunities to learn important facts which have just been discovered, they serve also to inspire one to renewed efforts, and to extend and deepen friendships between those who are working along various scientific lines. The latter desirable ends are greatly promoted by numerous dinners and smokers as well as by the fact that a large number of those who attend select the same hotel.

The weekly journal called *Science* is the official organ of the Association and the reader is referred to its columns for reports of the meetings of all sections. In some cases, more technical details, as regards the scientific results, may be found in the journals published by the various scientific societies which meet in affiliation with the Association. The next meeting of the Association is to be held at Atlanta, Georgia, under the presidency of Professor E. B. WILSON, Columbia University. Director Frank Schlesinger, Allegheny Observatory, and Professor F. R. Moulton, University of Chicago, were elected chairman and secretary respectively, of Section A. The term of the former office is only one year while that of the latter is five years.

G. A. MILLER,
Retiring Secretary of Section A.

MAXIMUM PARCELS UNDER THE NEW PARCEL POST LAW.

The size of a parcel that can be mailed under the rules of the Parcel Post Law is limited in three ways. It must be not more than three feet and six inches in length. It must not weigh more than eleven pounds. The sum of its length and girth must not exceed six feet. The girth is measured around a cross section perpendicular to the length; and since the greatest girth is the one measured, all such cross sections should be equal to the largest of them if it is desired that the volume be a maximum. Furthermore, since the circle has a larger area for a given perimeter than any other closed curve, the parcel should be in the form of a right circular cylinder if the volume is to be a maximum. Let R denote the radius of the base of the cylinder, V its volume and x its altitude or length. Then $V = \pi R^2 x$, and $2\pi R + x = 6$. From these two equations it is easily found by differential calculus that for a maximum value of V the length must be two feet and the girth four feet. The maximum value of V is 2.546 cubic feet. This is the largest mailable volume.

If the parcel to be mailed has its cross sections all equal and of a specified shape which is not circular, it is still true that the length should be two feet and

the girth four feet for maximum volume. To prove this, let P denote the perimeter of a cross section and A its area. As P changes in value, the shape of the cross section remaining constant, A varies as P^2. That is, $A = kP^2$. Then if L denote the length of the parcel, and V its volume, $V = kP^2L$ and $L + P = 6$. From these two equations the value of P that makes V a maximum is found to be 4. The corresponding value of L is 2.

Since a square has a larger area than any other rectangle having the same perimeter, the largest mailable box in the form of a rectangular parallelopiped is therefore one whose dimensions are $1 \times 1 \times 2$ feet. Its volume is 2 cubic feet. The largest mailable cube is one whose edge is 14.4 inches long. Its volume is 1.728 cubic feet. The largest mailable sphere has a diameter of 17.38 inches. Its volume is 1.592 cubic feet.

A cube whose edge is 15 inches is not mailable because the sum of its length and girth is 75 inches. But a right circular cylinder, whose length is 24 inches and whose diameter is 15.27 inches, is mailable, because the sum of its length and girth is only 72 inches, in spite of the fact that its volume exceeds that of the cube by more than 1,000 cubic inches, and the fact that it would take up more room in a mail car. Thus the use of the sum of length and girth as a measure of size leads to the absurd result that sometimes the smaller of two parcels will be rejected by the postal authorities because it is too large, while the larger of the two will be accepted as being small enough. In spite of this absurd result, the rule can be amply justified on the ground of the ease with which it can be applied. With a six-foot tape, the postal clerk can tell in a moment whether or not a parcel is too large to be mailed. The man who is waiting his turn at the post office window will be glad that the clerk does not have to compute the cubical content of every parcel presented.

W. H. BUSSEY.

NEW BOOKS.

W. H. BUSSEY, CHAIRMAN OF THE COMMITTEE.

A SHORT ACCOUNT OF THE HISTORY OF MATHEMATICS, 5th edition, by W. W. R. Ball. London and New York, Macmillan, 1912. 24 + 536 pp. $3.25.

PRACTICAL GEOMETRY AND GRAPHICS; a text book for students in technical and trade schools, evening classes, and for engineers, artisans, draughtsmen, architects, etc., by E. L. Bates and F. Charlesworth. New York, Van Nostrand, 1912. 8 + 621 pp. $2.00.

PRACTICAL MATHEMATICS, by E. L. Boter and F. Chorlesworth. New York, Van Nostrand, 1912. 8 + 513 pp. $1.50.

ELEMENTS OF DESCRIPTIVE GEOMETRY, by G. F. Blessing and L. A. Darling. New York, Wiley, 1912. 14 + 219 pp. $1.50.

THE METHOD OF ARCHIMEDES RECENTLY DISCOVERED BY HEIBERG; a supplement to "The works of Archimedes," 1897, edited by T. L. Heath. New York, Putnam, 1913. 51 pp. $0.75.

AN INTRODUCTION TO MATHEMATICAL PHYSICS, by R. A. Houston. New York, Longmans, 1913. 9 + 199 pp. $2.00.

MEMORANDA MATHEMATICA; a synopsis of facts, formulæ, and methods in elementary mathematics, by the head master of the Kingswood School, Bath, England. Oxford, University Press, 1912. 4 + 272 + 28 pp. $1.75.

LIFE ASSURANCE PRIMER; a textbook dealing with the practice and mathematics of life insurance, for advanced schools, colleges, and universities. 3d edition, revised and enlarged, by H. Moir. New York, Spectator Co., 1912. 7 + 230 pp. $2.00.

PRACTICAL MATHEMATICS; being the essentials of arithmetic, geometry, algebra, and trigonometry, in 4 parts. Parts 1, 2, by C. I. Palmer. New York, McGraw-Hill Co., 1912. $0.75 each.

GEOMETRICAL OPTICS, by A. S. Percival. New York, Longmans, 1913. 6 + 132 pp. $1.50.

A COURSE IN THE PRINCIPLES OF MECHANICAL DRAWING, by J. B. Whitmore. Columbus, O., Champlin Press, 1913. 2 + 2 + 72 pp. $1.85.

RELIABLE INTEREST TABLES, by E. V. Williams and G. G. Bodeen. Fresno, Cal., Bankers Novelty Co., 1912. $3.50.

BOOK REVIEWS.

New Analytic Geometry. By PERCEY F. SMITH, and ARTHUR SULLIVAN GALE. Ginn and Company, Boston, 1912. x+342 pages.

This new book on analytic geometry is considerably smaller than the *Elements of Analytic Geometry* by the same authors. Much of it follows, word for word, the old book, but many parts have been rewritten and rearranged. The book has been made smaller by shortening the work on the conic sections and by omitting the chapters on euclidean transformations, inversion, and poles and polars. The conics are defined separately, the parabola in terms of focus and directrix, and the ellipse and hyperbola in terms of their foci alone. They are studied by means of their equations in rectangular coordinates rather than by means of polar coordinates as was done in the beginning of the chapter on conics in the old book.

A special feature of the book is the new chapter on "Empirical Equations." The finding of equations to represent, at least approximately, curves given by experiment is not discussed in the current books on analytic geometry, although it is one of the most important parts of the subject for those who use analytic geometry in physics, experimental psychology, and statistics. Eleven pages are given to it in this book.

Another new chapter is on "Functions and Graphs." It is mostly a study of maxima and minima of functions without the use of the calculus. Practical problems in maxima and minima are given to be solved experimentally by means of carefully drawn graphs.

The work on transcendental curves and equations in rectangular coordinates is more extensive than in the old book and has been made into a separate chapter. It includes the graphical solution of transcendental equations, and some tables to lighten the work of computation. The authors say truly in the preface that a student loses interest in a function if he cannot calculate rapidly its numerical values. It is for this reason that they have put in the first chapter tables of squares and cubes, square and cube roots, and three-place tables of logarithms and trigonometric functions.

The book is considerably larger than the *Introduction to Analytic Geometry* by the same authors.

W. H. BUSSEY.

Mathematical Recreations and Essays. Fifth Edition. By W. W. ROUSE BALL. Macmillan and Co., London, 1911. xvi+492 pages.

A great deal of new matter has been added to this interesting book since it was first published in 1892. The fifth edition contains almost 250 pages more than the first and about 100 pages more than the fourth. The work on "Kirkman's School-Girls Problem" has been enlarged and made into a separate chapter. There is a paragraph on the same problem as proposed independently by J. Steiner in a somewhat more general form. There is a new chapter of 20 pages on "The Parallel Postulate," and one of 6 pages on the "Insolubility of the Algebraic Quintic." Those who amused themselves in their youth by making figures known as *Cat's Cradles* by twisting on the hands a loop of string will be interested in the new chapter on "String Figures." The subject is more extensive than most people think. The chapter is 32 pages long and is not supposed to be a complete discussion. It is only indirectly connected with mathematics. The author explains the presence of it and the older chapters on "Astrology" and "Ciphers" by saying that he deliberately gave the book a title which would allow him a free hand to write what he liked.

The parts of the book which are not new have been revised. In the chapter on mechanical recreations, after a discussion of the cut on a tennis ball and the spin on a cricket ball, the author has put in a paragraph on the flight of golf balls. In the chapter on matter and ether theories, he has added a page on the principle of relativity. These are typical instances of the way in which the book has been brought up to date.

W. H. BUSSEY.

An Elementary Treatise on Cross-Ratio Geometry, with Historical Notes. By Rev. JOHN J. MILNE. Cambridge University Press, 1911. xxiii+288 pp.

It is well known that our literature on secondary mathematics is too limited. The ambitious teacher of secondary mathematics, who reads English only, does not possess as good facilities for broadening his knowledge as do his German and French colleagues. Recently there has been considerable improvement along this line, and the volume before us is another step in the right direction.

The author is a classical scholar as well as a mathematician, and he has enriched his work by many historical notes which tend to make it much more attractive and helpful to teachers of mathematics. For the most part these notes are not reproduced from well-known histories, as is too often the case, but they generally bear the impress of originality and of independent study. This feature enhances greatly their value and their attractiveness. The same impression is gained in regard to other parts of the work, making the book valuable to the scholar as well as to the student who may approach it without a knowledge even of the meaning of cross-ratio.

Many readers might have preferred the term *Anharmonic Ratio* instead of *Cross-Ratio* in the title, as the former is more widely used. For instance, it is used almost exclusively in France, and it is in common use in Germany, Italy, and in the English-speaking countries. In view of these facts, we may perhaps not agree with the footnote on page 2, which states that the term cross-ratio "is now generally adopted," unless this reference is supposed to apply to Great Britain only. The term anharmonic ratio is not free from objection, since it is sometimes used in two senses—as including or as excluding harmonic ratio.

In these days of emphasis on things which are closely related to the larger modern mathematical theories, it becomes especially interesting to meet a work in which the emphasis is placed on closer contact with the old Greek mathematicians, and with their methods. It is true that even in the present work attention is frequently directed to the great improvements which have been made regarding methods employed by the Greeks in the use of anharmonic ratios, but these improvements are generally in line with Greek methods.

This close contact with the spirit of the Greeks has its disadvantages. For instance, the treatment of the matter on pages 5 and 9 would have gained considerably, both in clearness and in generality, if elementary notions of groups had been introduced; showing that we are dealing here with the operations of subtracting from unity and of dividing unity, and that the successive steps of this kind give rise to six distinct operations, which constitute a very common group. It might also have been observed that these operations are special cases of more general operations which give rise to the same group.[1]

By exhibiting such contact and by putting more emphasis on the underlying abstract notions it is not only possible to make the presentation more interesting for the mature student but it is also possible to prepare the way for further advances. The days of the special geometric methods of the Greeks are passing, and the more powerful and more comprehensive methods of modern mathematics are rightly coming more and more to the front. It is, however, true that the consistent use of such a fundamental concept as that of anharmonic ratio is a great step in advance of the older and more special geometric methods of the Greeks.

Almost half of the book, Chapters I–X, is devoted exclusively to a treatment

[1] Cf. "Groups of Subtraction and Division," *Quarterly Journal of Mathematics*, vol. 37 (1906), p. 80.

of the point and a straight line, and in this part the reader is not assumed to have any knowledge of geometry beyond the fundamental properties of similar triangles and ratio. The teacher of elementary geometry will therefore find no trouble in reading this part, and he cannot fail to derive from such a reading many new points of view and a deeper sense of the richness of geometry and the beauty of geometric results.

The second part of the book, beginning with Chapter XI, page 130, is devoted to a study of conic sections. A few elementary theorems in geometrical conics are here assumed, as the author believes that time can be saved by getting a working knowledge of the elements of conic sections from the ordinary textbooks before the student is introduced to the theorems of Pascal, Brianchon, Desargues, and others. Figures are given with almost all the theorems and many full solutions are worked out in the text.

Very little use is made of projection and the principle of duality is avoided, as the author desired to exhibit the use of the theory of anharmonic ratio by the direct demonstration of correlative theorems. The book is provided with a somewhat full table of contents, and also with a good index. A number of blank pages for notes are found at the end of the volume. As a whole, the book can be regarded as a thoroughly good piece of work and one which should be in the hands of all teachers of elementary geometry.

G. A. MILLER.

PROBLEMS AND QUESTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

ALGEBRA.

383. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

Find accurately to 6 decimals, $1.000000854^{1233451}$.

384. Proposed by H. C. FEEMSTER, York, Neb.

A man addressed n envelopes and wrote n checks in payment of n bills. Show that the number of ways of enclosing within each envelope one bill and one check in such a manner that in no instance all the enclosures shall be correct is

$$n!\left\{n! - \frac{(n-1)!}{1!} + \frac{(n-2)!}{2!} - \cdots + (-1)^n \frac{0!}{n!}\right\},$$

taking $0! = 0$.

385. Proposed by J. F. LAWRENCE, Stillwater, Okla.

Show that, if p is prime and > 3, $(2p)! - 2(p)!(p)!$ is divisible by p^3.

386. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given the sequence, $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \cdots, u_1, u_2, u_3, \cdots$.

Show that

$$\left.\frac{\sum_1^n u_i}{n}\right]_{n=\infty} = \frac{1}{2}.$$

GEOMETRY.

412. Proposed by S. LEFSCHETZ, The University of Nebraska.

To inscribe in a given circle an isosceles triangle in which the base plus the altitude is equal to a given length.

413. Proposed by D. F. KELLY, New York City.

To construct a triangle, having given the base, vertical angle and the ratio of the altitude to the difference of the other two sides.

414. Proposed by H. C. FEEMSTER, York, Neb.

To construct the cyclic quadrilateral, having given its four sides.

CALCULUS.

334. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Solve the differential equation:

$$\frac{\partial^2 T}{\partial u \partial v} + \frac{2v}{u^2+v^2+1}\frac{\partial T}{\partial u} + \frac{2u}{u^2+v^2+1}\frac{\partial T}{\partial v} = 0.$$

335. Proposed by W. R. LEBOLD, Cambridge, Ohio.

Let $\rho = F(\theta, \phi)$ be the equation in polar coordinates of a closed surface. Show that the volume of the solid bounded by the surface is equal to the double integral.

$$\tfrac{1}{3}\int\int \rho \cos\gamma \, d\sigma$$

extended over the whole surface, where $d\sigma$ represents the element of area, and γ the angle which the radius vector makes with the exterior normal.

[Goursat-Hedrick, *Analysis*, p. 325, ex. 9.]

336. Proposed by EVA S. MAGLOTT, Ada, Ohio.

If a right cone stands on an ellipse, prove that its superficial area is

$$\frac{\pi}{2}(OA + OA')(OA \cdot OA')^{\frac{1}{2}} \sin\alpha,$$

where O is the vertex of the cone, A and A' the extremities of the major axis of the ellipse, and α is the semi-angle of the cone.

MECHANICS.

271. Proposed by B. F. FINKEL, Springfield, Mo.

A hollow spherical shell is filled with a frictionless fluid and rolls down a rough inclined plane. After rolling t seconds, the fluid suddenly solidifies. Determine the subsequent motion of the spherical shell.

272. Proposed by J. F. LAWRENCE, Stillwater, Okla.

A perfectly rough circular cylinder is fixed with its axis horizontal. A sphere is placed on it in a position of unstable equilibrium, and projected with a given velocity parallel to the axis of the cylinder. If the sphere be slightly disturbed in a horizontal direction perpendicular to the direction of the axis of the cylinder, determine at what point the sphere will leave the cylinder.

273. Proposed by F. P. MATZ, Reading, Pa.

A person is placed on a perfectly smooth surface. How may he get off?

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

187. Proposed by E. T. BELL ,New York, N. Y.

If m is any integer, P the product of all the distinct prime factors of m and λ their number, and if $N(x)$ denote the number of divisors of x, then

$$6^{\lambda}\Sigma N(d) \cdot N\left(\frac{m}{d}\right) = N(m) \cdot N(Pm) \cdot N(P^2m),$$

where the sign of summation extends to all divisors of m.

188. Proposed by ELMER SCHUYLER, Brooklyn, New York.

Solve the congruence, $3x^3 + 4x + 5 \equiv 0$, mod. 20 [Reid].

189. Proposed by V. M. SPUNAR, Chicago, Ill.

If p and $p_1 = 2^p - 1$ are primes, then are the numbers of the sequence, $p_1 = 2^p - 1$, $p_2 = 2^{p_1} - 1$, $p_3 = 2^{p_2} - 1$, $\cdots$, $p_n = 2^{p_{n-1}} - 1$ all primes?

SOLUTIONS OF PROBLEMS.

ALGEBRA.

379. Proposed by C. N. SCHMALL, New York, N. Y.

Given $y = ax + b$. If the values of x increase in an arithmetical progression, show that the values of y vary likewise.

SOLUTION BY T. M. BLAKSLEE, Ames, Iowa.

We have

$$y_1 = ax_1 + b, \quad y_2 = ax_2 + b, \quad y_3 = ax_3 + b, \quad \cdots.$$

Hence,

$$y_2 - y_1 = a(x_2 - x_1), \quad y_3 - y_2 = a(x_3 - x_2), \quad \cdots.$$

Since $x_1, x_2, x_3, \cdots, x_n$ are in an arithmetical progression,

$$x_2 - x_1 = x_3 - x_2 = x_4 - x_3, \quad \cdots, \quad = d \text{ say.}$$

Hence,

$$y_2 - y_1 = y_3 - y_2 = y_4 - y_3, \quad \cdots, \quad = ad.$$

Hence,

$$y_2 = y_1 + ad, \quad y_3 = y_2 + ad = y_1 + 2ad,$$

$$y_4 = y_3 + ad = y_1 + 3ad, \quad \cdots, \quad y_n = y_1 + (n-1)ad,$$

an arithmetical progression whose common difference is ad.

Also solved by A. H. Holmes, A. M. Hardin, Elmer Schuyler, W. T. Risley, T. J. Fitzpatrick, and the Proposer.

380. Proposed by J. K. ELWOOD, Lucas, Kansas.

A and B set out to walk around a cinder path a mile in circumference, and walk 3 hours, A walking 8 miles farther than B. Each reduced his rate one mile per hour at the end of the first hour, and again one mile per hour at the end of the second hour, his speed being otherwise uniform. They start in the same direction, but 12 minutes after A has passed B the third time he turns and walks in the other direction until 6 minutes after he has met B the third time, when he returns to his original direction, and overtakes B four times more. Determine their initial velocities.

SOLUTION BY THE PROPOSER.

A gains $2\frac{2}{3}$ miles an hour, or one mile in 22.5 minutes. He overtakes and passes B the first three times in 67.5 minutes, and the last three times in 67.5 minutes. All of A's travel requiring calculation, therefore, occurs in

$$180 - (67.5 + 67.5) = 45 \text{ minutes of the second hour.}$$

Let $x =$ the sum of their rates during second hour.

In the 12 minutes after passing B the third time A gains 8/15 mile. Hence, after A reverses his direction, the three meetings will require them to travel $2\frac{8}{15}$ miles at x miles an hour, or $x/60$ mile per minute. Then $2\frac{8}{15} \div x/60 = 152/x$ is the number of minutes required for the three meetings.

After this they continue in opposite directions 6 minutes, and together travel $x/10$ miles.

Case I.—If they do not meet during the 6 minutes, then A, after resuming his original direction, must first gain $x/10$ miles to overtake B, which requires $x/10 \times 22.5$ minutes. Hence

$$12 + \frac{152}{x} + 6 + \frac{x}{10} \times 22.5 = 180 - (67.5 + 67.5) = 45 \text{ minutes.}$$

or,

$$\frac{9x}{4} + \frac{153}{x} = 27, \tag{1}$$

from which $x = 18 \pm \sqrt{-284}$.

Case II.—If they meet once during the 6 minutes, then A must gain $(x/10 - 1)$ miles, which he will do in $(x/10 - 1) \times 22.5$ minutes.

Then equation (1) becomes

$$\frac{9x}{4} - 22.5 + \frac{152}{x} = 27, \tag{2}$$

from which $x = 18.31$.

Hence A's rate is now 10.49 miles per hour and B's is 7.82 miles per hour. Then A began walking at 11.49 miles per hour and B at 8.82 miles per hour. In this way a complete time-table has been constructed.

GEOMETRY.

407. Proposed by S. LEFSCHETZ, University of Nebraska.

To construct a right triangle knowing the sum of the sides of the right angle and the sum of one of them and the hypothenuse.

I. Solution by A. H. Holmes, Brunswick, Maine.

Let a be the sum of the two sides and b be the sum of one side and the hypothenuse of the required triangle. Draw $AB = 2a$ and extend to C, making $BC = b - a$. On AC describe the semi-circle, ACD. At B erect a perpendicular BD cutting the circumference in D. On DB take $DE = BC$. Then BE is one of the required sides. From O, the middle point of AB, lay off $OF = BE$. Then BF is the other side and BEF is the required triangle. For, letting x and y represent the sides, we have $x + y = a$ and $x + \sqrt{x^2 + y^2} = b$, from which $x = \sqrt{2b(b-a)} - (b - a)$ and $y = b - \sqrt{2b(b-a)}$.

II. Solution by T. M. Blakslee, Ames, Iowa.

Given $b + a$ and $a + h$, b, a, and h being the base, altitude, and hypothenuse respectively. Draw two parallel rays, l and s at distance $a + h$. On l lay off $AD = b + a$. Draw DE making $\angle ADE = 45°$. The ray of DE contains the vertex B of the required triangle ACB. This vertex is also on the parabola having s as directrix and A as focus.

Having found B, draw BC perpendicular to l and join A and B.

Excellent solutions were also received from A. M. Harding, C. N. Schmall, H. C. Feemster, Levi S. Shively, Elmer Schuyler, W. T. Risley, and Francis C. Rust.

408. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given a point A on a circle and a chord of the circle; to draw a chord through A so that it shall be bisected by the given chord.

Solution by A. M. Harding, University of Arkansas.

Let C be the center of the circle. On CA as a diameter describe a circle. This circle will cut the chord, in general, in two points P_1, P_2. Join A to P_1 and A to P_2. Then these are the required chords. If A be on the minor arc of the circle there will be two solutions. If A be on the major arc there will be two solutions, one solution, or no solution according as d is less than, equal to, or greater than $r - c$, where d is the distance from A to the given chord, c is distance from center to the given chord and r is the radius of given circle.

Also solved by T. M. Blakslee, C. N. Schmall, A. H. Holmes, Levi S. Shively, Francis C. Rust, G. W. Hartwell, W. R. Lebold, H. C. Feemster, and Elmer Schuyler.

CALCULUS.

326. Proposed by C. N. SCHMALL, New York City.

Prove that

$$\int_0^\infty \frac{(\tan^{-1} ax)^2 - (\tan^{-1} bx)^2}{x}\, dx = \frac{\pi^2}{4}(\log a - \log b).$$

Solution by the Proposer.

Take the integral,

$$u = \int_0^\infty [f(x) - f(kx)]\frac{dx}{x}. \tag{1}$$

Substitute for x the successive values kx, k^2x, k^3x, $\cdots$, k^nx.

We then have

$$u = \int_0^\infty [f(kx) - f(k^2x)]\frac{dx}{x}, \tag{2}$$

$$u = \int_0^\infty [f(k^2x) - f(k^3x)]\frac{dx}{x}, \tag{3}$$

. . . . . .

$$u = \int_0^\infty [f(k^{n-1}x) - f(k^nx)]\frac{dx}{x}. \tag{n}$$

Adding these equations, we get

$$nu = \int_0^\infty [f(x) - f(k^n x)] \frac{dx}{x} = \int_0^\infty [f(x) - f(rx)] \frac{dx}{x},$$

where $r = k^n$.

Now, suppose n to become infinite, r remaining constant, then k will approach unity as a limit, and we have

$$\int_0^\infty [f(x) - f(rx)] \frac{dx}{x} = n \int_0^\infty [f(x) - f(r^{1/n} x)] \frac{dx}{x}.$$

The right member of this equation (in the limit) reduces to the indeterminate form 0/0. Differentiating in this form with respect to $1/n$, we get

$$-\int_0^\infty f'(x) \log r dx = \log r[f(0) - f(\infty)].$$

Now in the given integral, $f(x) = (\tan^{-1} ax)^2$. Hence,

$$f(0) = 0, \qquad f(\infty) = \tfrac{1}{4}\pi^2;$$

also,

$$r = \frac{b}{a}.$$

Hence the integral is equal to $\frac{1}{4}\pi^2 (\log a - \log b)$.

This problem may be readily solved by Frullani's Theorem, Williamson's *Integral Calculus*, page 155. Professor J. Scheffer solved the problem by means of this Theorem, but failed to notice that in the above problem $\phi(ax) = [\tan^{-1} ax]^2$ instead of $\tan^{-1} ax$. Ed. F.

327. Proposed by RICHARD P. LOCHNER, Philadelphia, Pa.

A hound is at the middle point of the side of a square field and a fox is at an adjacent corner. How far will the hound run to catch the fox if the fox runs on the perimeter of the field and the hound runs directly towards the fox at all times, the hound running n times as fast as the fox. Where will the race end?

No solution of this problem has been received.

328. Proposed by M. E. GRABER, Tiffin, Ohio.

Prove that

$$\frac{E_m}{\pi/2m} \int_0^{\pi/2m} \left[\sin a + \sin\left(\frac{\pi}{m} + a\right) + \cdots + \sin\left(\pi \frac{m-1}{m} + a\right) da \right] = \frac{2mE_m}{\pi}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

From trigonometry we have

$$\sin a + \sin\left(\frac{\pi}{m} + a\right) + \cdots + \sin\left(\pi \frac{m-1}{m} + a\right) = \frac{\sin\left(a + \dfrac{m-1}{2m} \cdot \pi\right)}{\sin \dfrac{\pi}{2m}}.$$

Hence the given integral takes the form

$$\frac{E_m}{\frac{\pi}{2m}} \cdot \frac{1}{\sin\frac{\pi}{2m}} \int_0^{\pi/2m} \sin\left(a + \frac{m-1}{2m} \cdot \pi\right) da$$

$$= \frac{2mE_m}{\pi} \cdot \frac{1}{\sin\frac{\pi}{2m}} \cdot \left[-\cos\left(a + \frac{m-1}{2m}\pi\right)\right]_0^{\pi/2m}$$

$$= \frac{2mE_m}{\pi} \cdot \frac{1}{\sin\frac{\pi}{2m}} \cdot \sin\frac{\pi}{2m} = \frac{2mE_m}{\pi}.$$

Solved similarly by Geo. W. Hartwell and J. Scheffer.

329. Proposed by C. N. SCHMALL, New York City.

Show that the general differential equation

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P^2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_{n-1}\frac{dy}{dx} + P_n y = V,$$

where $P_1, P_2, \cdots P_n, V$ are known functions of x, has a solution in the form

$$y - v_1 \int v_2 dx \int v_3 dx \cdots \int \frac{V dx}{v_1 v_2 v_3 \cdots v_n}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

Let $y = v_1 z_1$. Then the given differential equation becomes

$$v_1\frac{d^n z_1}{dx^n} + Q_1\frac{d^{n-1}z_1}{dx^{n-1}} + \cdots + Q_{n-1}\frac{dz_1}{dx}$$

$$+ z_1\left[\frac{d^n v_1}{dx^n} + P_1\frac{d^{n-1}v_1}{dx^{n-1}} + \cdots + P_{n-1}\frac{dv_1}{dx} + P_n v_1\right] = V. \quad (1)$$

Now suppose the expression in brackets, when equated to zero, is satisfied by v_1. In that case this expression will vanish, and we have left a linear equation of the $(n-1)$th degree in dz_1/dx.

Putting then

$$\frac{dz_1}{dx} = y_1, \quad (2)$$

and $Q_r = v_1 R_r$, we have

$$\frac{d^{n-1}y_1}{dx^{n-1}} + R_1\frac{d^{n-2}y_1}{dx^{n-2}} + \cdots + R_{n-1}y_1 = \frac{V}{v_1}.$$

Now this equation has the same form as the original equation and is one degree lower. Proceeding in the same way we at last get

$$y_n = \frac{V}{v_1 v_2 v_3 \cdots v_{n-1} v_n},$$

but

$$y = v_1 z_1 = v_1 \int y_1 dx,$$

by (2). Similarly,

$$y_1 = v_2 z_2 = v_2 \int y_2 dx,$$

. . . . . .

$$y_{n-1} = v_n z_n = v_n \int y_n dx.$$

Hence

$$y = v_1 \int v_2 dx \int v_3 dx \cdots \int \frac{V dx}{v_1 v_2 v_3 \cdots v_{n-1} v_n}.$$

Also solved in the same manner by the Proposer.

MECHANICS.

263. Proposed by C. N. SCHMALL, New York City.

A railroad car is rounding a curve of radius r with a velocity v, $2d$ being the distance between the rails. If h is the distance of its center of gravity above the rails, g having the usual meaning, show that the weight of the car is divided between the outer and inner rails in the ratio $\frac{dgr + v^2h}{dgr - v^2h}$.

I. SOLUTION BY H. PRIME, Boston, Mass.

Let R and R' be the reaction of the weight on the outer and inner rails respectively. Then we have for the moments

$$\text{about inner rail,} \qquad 2dR = dmg + \frac{hmv^2}{r};$$

and

$$\text{about outer rail,} \qquad 2dR' = dmg - \frac{hmv^2}{r}.$$

Whence

$$\frac{R}{R'} = \frac{dgr + v^2h}{dgr - v^2h}.$$

II. SOLUTION BY J. SCHEFFER, Hagerstown, Md.

Denoting the pressures upon the rails by P and Q, and the angle which a plumb-line makes with the vertical by ϕ, the vertical strikes the distance between the rails at a point distant $d - h \tan \phi$ and $d + h \tan \phi$ from the rails respectively. Hence $P(d - h \tan \phi) = Q(d + h \tan \phi)$. Then

$$\frac{P}{Q} = \frac{d + h \tan \phi}{d - h \tan \phi}.$$

But $\tan \phi = v^2/rg$, and

$$\frac{P}{Q} = \frac{dgr + hv^2}{dgr - hv^2}.$$

No solution of 264 has been received. We shall be pleased to receive solutions of unsolved problems as well as a variety of good problems in the several problem departments.

NEWS AND NOTES.

FLORIAN CAJORI, CHAIRMAN OF THE COMMITTEE.

New subscribers to the MONTHLY are invited to read the reflections of Editor Finkel, in the December issue, on the nineteen years' history of this journal.

The spring meeting of the Chicago Section of the American Mathematical Society will be held at the University of Chicago, on Friday and Saturday March 21, 22.

At the annual meeting of the American Mathematical Society in Cleveland, Professor E. B. VAN VLECK, of the University of Wisconsin, was elected president. The next summer meeting of the Society, including a colloquium, will be held at Madison, Wis., early in September, 1913.

We notice with regret the death of Professor WILHELM FIEDLER of the Polytechnicum in Zürich, known for his work in geometry and his translations into German of the books of George Salmon.

In this country have occurred the deaths of OLIVER CLINTON WENDELL, professor of astronomy at Harvard University, of EBEN LOOMIS who was for half a century connected with the Nautical Almanac Office, and of EDWIN SMITH who was connected with the U. S. Coast and Geodetic Survey since 1870.

The first part of EULER's *Commentationes* on elliptic integrals has just appeared as Vol. XX of the first series of Euler's *Opera Omnia*.

Professor JACQUES HADAMARD has been elected a member of the Paris Academy of Sciences in the section of geometry, in succession to the late HENRI POINCARÉ.

On January 19, 1913, occurred in Denver the death of Robert Gauss, who had been long connected with the *Denver Republican*. On the same day died also his brother, Charles H. Gauss, of St. Louis. They were sons of Eugene Gauss, and grandsons of the mathematician, CARL FRIEDRICH GAUSS.

Announcement was made of the death, on December 7, of SIR GEORGE HOWARD DARWIN, Plumian professor of astronomy and experimental philosophy at the University of Cambridge. He was a leading authority on tidal phenomena. He was president of the fifth international congress of mathematicians which was held in Cambridge University, August 21 to 28, 1912. In his address before the Congress he spoke of the Cambridge School of applied mathematics and its work during the last century.

"The Cambridge School of Mathematics" is the title of an historical article by W. W. ROUSE BALL in the *Mathematical Gazette* for July, 1912. Without entering much into critical and philosophical inquiries, Ball tells in a pleasant way of mathematical research and teaching at Cambridge from the time of Tonstall, Recorde and John Dee to the present time. This number of the *Gazette* is

adorned by fine page portraits of Sir Isaac Newton, Sir George Darwin, Sir Joseph Larmor, Professor E. W. Hobson and Professor A. E. H. Love.

The collected papers of SOPHUS LIE are to be published by the firm of B. G. Teubner in seven volumes, under the editorship of Friedrich Engel. Subscription prior to April 1, 1913, is at the reduced rate of about 160 marks. The first two volumes will be on geometry, the next two on differential equations, the two following on transformation groups, while the last volume will be reserved for the publication of residual manuscripts.

At the Cleveland meeting of the American Association for the Advancement of Science held last December, Professor F. R. MOULTON, of the department of astronomy at the University of Chicago, was elected secretary of Section A, to succeed Professor G. A. Miller, of the University of Illinois, who had held this position since 1907.

After this year, mathematics is to be an elective subject in the College of Science, Literature, and Arts at the University of Minnesota. For the last five years, college algebra and trigonometry have been required of those students who entered the college without having had both higher algebra and solid geometry. Before that time, one year of work in college mathematics was required of all candidates for the Bachelor of Arts degree.

A most interesting account of HENRI POINCARÉ and his work, from the pen of Professor F. R. Moulton, appeared in the December number of *Popular Astronomy*. Another sketch which touches a different phase of the work of Poincaré appeared in *Nature* (London) of November 28, 1912.

The University of Oregon has begun the publication of a monthly, called the *Extension Monitor*. In the first number (January, 1913) Professor E. E. DE COU gives a brief account of the work of the international commission for the improvement of mathematical teaching.

Professor W. E. BYERLY, of Harvard University, will become professor emeritus at the close of the present academic year. He is well known for his texts on the calculus and on Fourier's series and spherical harmonics, as well as for his unusual ability as a lecturer and teacher.

Isis is the name of a new international journal to be devoted to the history of science, under the editorship of GEORGE SARTON, D.Sc., of Wondelgem-lez-Gand in Belgium. On the "Comité de patronage" are 30 names, including those of two Americans, JACQUES LOEB, of the Rockefeller Institute for Medical Research, and DAVID EUGENE SMITH, of Columbia University.

The Royal Society of Edinburgh is preparing to celebrate the tercentenary of the first important British contribution to mathematical science, namely the invention of logarithms by NAPIER of Merchiston in 1614. What form the cele-

bration will take is not yet announced, but probably a bird's eye view of mathematical progress along the lines of algebra will be one of the features.

In the students' section of the mathematics club of the University of Illinois the monthly programs for the present academic year include the following papers: Errors in mathematical literature, by Professor G. A. Miller; Construction of regular polygons, by E. A. Kircher; The practical use of the slide rule, by C. R. Horrell; Mathematical induction, by C. C. Yen; The Hindu-Arabic numerals, by F. A. Murray; The calculation of π, by Miss Peach Andrews; The theory of the gyroscope, by Miss Mildred Seyster; Mathematical models, by Professor A. Emch.

Professor Favaro, of Padua, at the October meeting of the Italian society for the advancement of science, gave an account of an unpublished translation of the writings of ARCHIMEDES, found among the Galileoan manuscripts in the Library of Florence. The translation was made by Vincenzio Viviani, a pupil of Galileo, and was apparently intended for publication.

The courses in mathematics to be offered at the University of Chicago in the summer session, 1913, include the following: By Professor E. H. MOORE, Seminar on point set theory, Fourier's series, Trigonometry; by Professor OSKAR BOLZA, Linear integral equations, Theory of functions of a complex variable; by Professor F. R. MOULTON, Theory of differential equations, Descriptive astronomy; By Professor H. E. SLAUGHT, Differential equations, Integral calculus; by Professor J. W. A. YOUNG, Differential calculus, Critical review of secondary mathematics; by Professor E. J. WILCZYNSKI, Theory of equations, Analytic geometry; by Professor G. A. BLISS, Projective geometry, College algebra; by Professor A. C. LUNN, Applied mathematics, Graphical analysis; by Professor W. D. MACMILLAN, Celestial mechanics, Observational astronomy.

The United States Bureau of Education has just published a *Bibliography of the Teaching of Mathematics* covering the period from 1900 to 1912, by David Eugene Smith and Charles Goldziher. This Bulletin gives 1,849 titles of books and articles on the teaching of mathematics that have appeared since 1900. The Bulletin will be sent gratis upon application to the United States Commissioner of Education, Washington, D. C.

Some of our readers may not agree with all the conclusions reached by the author of "Minimum Courses in Engineering Mathematics" in this issue. It is needless to say that the Editors assume no responsibility for the views expressed in any signed article. They will welcome the expression of dissenting views in the interest of helpful discussion along this line.

Since the last issue, Northwestern University has been added to the list of institutions contributing to the subsidy fund of the MONTHLY, making the total number now ten universities and two colleges. Professor D. R. CURTISS becomes a member of the Editorial Board whose assistance will be greatly valued.

We are again unable to issue the MONTHLY on time, and it may require still another month for the printer to catch up with the schedule. This number is also larger than normal, though we shall be pleased if subscription returns may make it possible to continue this larger size.

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THE

AMERICAN

MATHEMATICAL MONTHLY

VOLUME XX MARCH, 1913 NUMBER 3

HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

III. THE CREATION OF A THEORY OF LOGARITHMS OF COMPLEX NUMBERS BY EULER.

1747–1749.

It is desirable to look back, for a moment, over the 35 years of history of logarithms of negative and complex numbers. Thus far only three mathematicians have attempted to unravel the mysteries of this subject, namely Leibniz, John Bernoulli I and Euler. Their discussions were carried on entirely by letter; these letters were not published at the time. No articles or memoirs on this controversy had reached the press. The question had not been brought to the attention of the mathematical public.

Publicity on this subject began in 1745. In that year was published the correspondence between Leibniz and John Bernoulli I.[1] The reading of that correspondence acted as a tremendous stimulus upon Euler. As a boy of 20 he himself, as we have seen, had corresponded on this subject with his revered master, John Bernoulli I. That correspondence had set bare serious difficulties of the subject, but had not removed them. Since that time he had discovered the exponential expressions for $\sin x$, $\cos x$, and $\cos x + i \sin x$; he had acquired a deeper insight into the properties of imaginary numbers. It was in 1745 that he completed his manuscript on the *Introductio*, which was issued from the press three years later. Two years later (in 1747) he carried on his classic researches on logarithmic theory. He discussed this subject in his correspondence with D'Alembert. Some of Euler's letters on this subject have been published only recently. For that reason not even Lampe's historical essay is based on a study of all the material that is now available. For convenience of reference we give Euler's writings on this subject, the year in which they were penned, and the year of publication. The first two publications we have already reviewed.

[1] *Virorum celeberr. Got. Gul. Leibnitii et Johan. Bernoullii commercium philosophicum et mathematicum. Lausannæ et Genevæ*, 1745.